Exercise 3C

1 a $v^2 = \omega^2\left(a^2 - x^2\right)$

$a = 0.5, \quad x = 0 \quad v = 2$

$2^2 = \omega^2 \times 0.5^2$

$\omega = \frac{2}{0.5} = 4$

$\text{period} = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}$

The period is $\frac{\pi}{2}$ s.

b $x = 0.2\,\text{m} \quad v^2 = 4^2\left(0.5^2 - 0.2^2\right)$

$v = 1.833...$

When $OP = 0.2\,\text{m}$ the speed of P is 1.83 m s^{-1} (3 s.f.)

2 a $\text{period} = \frac{2\pi}{\omega} = \frac{\pi}{3}$

$\therefore \omega = 6$

$v^2 = \omega^2\left(a^2 - x^2\right)$

$6^2 = 6^2\left(a^2 - 0^2\right)$ ← Maximum speed occurs when $x = 0$.

$\therefore a = 1$

The amplitude is 1 m.

b $x = a \sin \omega t$ ← The speed 0.3 s after passing O is required.

$v = a\omega \cos \omega t$ ← Differentiate the line above to obtain v.

$t = 0.3s \quad v = 1 \times 6\cos(6 \times 0.3)$

$v = 6\cos 1.8$

$v = -1.363$

When $t = 0.3$, P has speed 1.36 m s^{-1} (3 s.f.) ← Speed is positive.

3 a $v^2 = \omega^2(a^2 - x^2)$

$x = 0,\ v = 10\,\text{m}\,\text{s}^{-1} \quad 10^2 = \omega^2 a^2$ **(1)** ← Maximum speed occurs when $x = 0$.

$\ddot{x} = -\omega^2 x$

$x = -a, \ddot{x} = 20\,\text{m}\,\text{s}^{-2}$

$20 = +\omega^2 a$ **(2)**

(1) $\div$ **(2)** $\frac{100}{20} = +\frac{\omega^2 a^2}{\omega^2 a}$ ← Maximum acceleration occurs when $x = -a$.

$a = 5$

The amplitude is 5 m.

3 b Using **(1)** $10 = a\omega$

$$10 = 5\omega$$

$$\omega = 2$$

$$\text{period} = \frac{2\pi}{\omega} = \pi$$

The period is π s.

4 $\text{period} = \frac{2\pi}{\omega} = \frac{3\pi}{5}$

$$\omega = \frac{10}{3}$$

$$v^2 = \omega^2(a^2 - x^2)$$

$$v^2 = \left(\frac{10}{3}\right)^2\left(0.4^2 - 0\right)$$

Maximum speed occurs when $x = 0$.

$$v = \frac{10}{3} \times 0.4 = \frac{4}{3}$$

The maximum speed is $\frac{4}{3}$ m s^{-1}.

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$$\ddot{x} = -\omega^2 x$$

$\ddot{x} = 15\text{ m s}^{-2}, x = a$

$$15 = \omega^2 a \qquad \textbf{(1)}$$

$$v^2 = \omega^2(a^2 - x^2)$$

First find a and ω (see question 3.)

$v = 18\text{ m s}^{-1}, x = 0 \qquad 18^2 = \omega^2 a^2 \qquad \textbf{(2)}$

(2) ÷ **(1)** $\frac{18^2}{15} = \frac{\omega^2 a^2}{\omega^2 a}$

$$a = \frac{18^2}{15} = 21.6$$

Using **(2)** $a\omega = 18$

$$\omega = \frac{18}{21.6} = 0.8333...$$

$$v^2 = \omega^2(a^2 - x^2)$$

$$v^2 = 0.833...^2\left(21.6^2 - 2.5^2\right)$$

$$v = 17.87...$$

The speed is 17.9 m s^{-1} (3 s.f.)

6 a period $= \dfrac{2\pi}{\omega} = \dfrac{\pi}{2}$

$\omega = 4$

$v^2 = \omega^2(a^2 - x^2)$

$x = 1.2\text{ m} \quad v = 1.5\text{ m s}^{-1}$

$1.5^2 = 4^2(a^2 - 1.2^2)$

$a^2 = \dfrac{1.5^2}{4^2} + 1.2^2$

$a = 1.257...$

The amplitude is 1.26 m (3 s.f.).

Use the period to find ω.

Then use $v^2 = \omega^2(a^2 - x^2)$ with $x = 1.2$ and $v = 1.5$ to find a.

b $x = a\sin\omega t$

$x = 1.26\sin 4t$

Use $x = a\sin\omega t$ as $x = 0$ when $t = 0$.

7 a period $= \dfrac{2\pi}{\omega} = \dfrac{1}{6}$

$\omega = 12\pi$

$v^2 = \omega^2(a^2 - x^2)$

$5^2 = (12\pi)^2(a^2 - 0)$

$a = \dfrac{5}{12\pi} = 0.1326...$

The amplitude is 0.133 m (3 s.f.)

The period is the time for one complete oscillation.

You are told the magnitude of the acceleration at A.

b $\ddot{x} = -\omega^2 x$

$20 = |-12^2\pi^2|x$

$x = \dfrac{20}{144\pi^2}$

$x = 0.01407...$

$OA = 0.0141\text{ m}\,(3\text{s.f.})$

8 a $v^2 = \omega^2(a^2 - x^2)$

$x = 0.6\,\text{m}, v = 3\,\text{m s}^{-1}$

$3^2 = \omega^2\left(a^2 - 0.6\right)^2$ **(1)**

$x = 0.2\,\text{m}, v = 6\,\text{m s}^{-1}$

$6^2 = \omega^2\left(a^2 - 0.2\right)^2$ **(2)**

(2) $\div$ **(1)** $\dfrac{6^2}{3^2} = \dfrac{\omega^2\left(a^2 - 0.2^2\right)}{\omega^2\left(a^2 - 0.6^2\right)}$

$4\left(a^2 - 0.6^2\right) = a^2 - 0.2^2$

$3a^2 = 4 \times 0.6^2 - 0.2^2$

$a^2 = \dfrac{4 \times 0.6^2 - 0.2^2}{3}$

$a = 0.6831...$

The distance *AB* is 1.37 m (3 s.f.)

AB is twice the amplitude.

b Using **(1)** $9 = \omega^2\left(0.6831^2 - 0.6^2\right)$

$\omega^2 = \dfrac{9}{(0.6831^2 - 0.6^2)}$

$\omega = 9.187$

$\text{period} = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{9.187} = 0.6838...$

The period is 0.684s (3 s.f.).

9 a $\text{period} = \dfrac{2\pi}{\omega} = 2\pi$

First find ω.

$\omega = 1$

$v^2 = \omega^2\left(a^2 - x^2\right)$

Now use $v^2 = \omega^2(a^2 - x^2)$ to find **a**.

$x = 1\,\text{m}, v = 0.1\,\text{m s}^{-1}$

$0.1^2 = 1^2\left(a^2 - 1^2\right)$

$a^2 = 0.1^2 + 1^2$

$a = 1.004...$

$v_{\max} = \omega a$

$= 1 \times 1.004...$

Maximum speed occurs when $x = 0$.

The maximum speed is 1.00 m s^{-1} (3 s.f.).

b $v^2 = 1\left(1.004^2 - 0.4^2\right)$

$v = 0.9219...$

The speed is 0.922 m s^{-1} (3 s.f.).

10 $a = \frac{2.5}{2} = 1.25$

$$\text{Period} = \frac{2\pi}{\omega} = \frac{60}{30} = 2$$

30 oscillations per minute $\Rightarrow$ 2s for 1 oscillation

$$\omega = \pi$$

$$v_{max} = a\omega$$
$$= 1.25 \times \pi$$

$$\text{maximum K.E.} = \frac{1}{2}mv_{max}^2$$
$$= \frac{1}{2} \times 1.2 \times 1.25^2 \times \pi^2$$
$$= 9.252...$$

The maximum K.E. is 9.25 J (3 s.f.).

11 a $a = 0.8 \div 2 = 0.4\,\text{m}$

The amplitude is half the distance between the highest and lowest points.

$$\text{period} = \frac{2\pi}{\omega} = 2$$
$$\omega = \pi$$
$$v^2 = \omega^2(a^2 - x^2)$$
$$x = 0 \quad v = \omega a$$
$$v = \pi \times 0.4$$
$$v = 1.256...$$

The maximum speed is 1.26 m s^{-1} (3 s.f.).

b 0.6 m from highest point

$$\Rightarrow x = -0.2\,\text{m}$$

The buoy is now below the centre.

$$x = a\cos\omega t$$

You want the time from the highest point.

$$-0.2 = 0.4\cos\pi t$$
$$\cos\pi t = -0.5$$
$$t = \frac{1}{\pi}\cos^{-1}(-0.5)$$
$$t = \frac{1}{\pi} \times \left(\pi - \frac{\pi}{3}\right)$$
$$t = \frac{2}{3}$$

The buoy takes $\frac{2}{3}$ s to fall 0.6 m.

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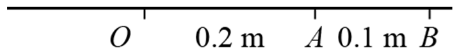


$$\text{period} = \frac{2\pi}{\omega} = 2$$

$$\therefore \omega = \pi$$

$$x = a\sin\omega t$$

$$x = 0.5\sin\pi t$$

$$x = 0.2\text{ m}\quad 0.2 = 0.5\sin\pi t_1$$

$$\pi t_1 = \sin^{-1}\left(\frac{0.2}{0.5}\right) = \sin^{-1}\left(\frac{2}{5}\right)$$

$$x = 0.3\quad \pi t_2 = \sin^{-1}\left(\frac{3}{5}\right)$$

$$\text{time } A \to B = t_2 - t_1$$

$$= \frac{1}{\pi}\left(\sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\frac{2}{5}\right)\right)$$

$$= 0.07384...$$

The time to move directly from A to B is 0.0738 s (3 s.f.).

> Use $x = a\sin\omega t$ to find the time to go from O to A and the time to go from O to B.

13 a

$$x = 4\sin 2t$$

$$\dot{x} = 8\cos 2t$$

$$\ddot{x} = -16\sin 2t$$

$$\ddot{x} = -4(4\sin 2t)$$

$$\ddot{x} = -4x$$

$\therefore$ S.H.M.

> Differential the given equation twice.

b amplitude = 4 m

$$\text{period} = \frac{2\pi}{2} = \pi\text{ s}$$

> Compare $x = 4\sin 2t$ with $x = a\sin\omega t$ to obtain a and w.

c

$$v^2 = \omega^2(a^2 - x^2)$$

$$x = 0\quad v^2 = 4(4^2 - 0)$$

$$v = 8$$

The maximum speed is 8 m s^{-1}.

d

$$x = 4\sin 2t$$

$$\dot{x} = 8\cos 2t$$

$$\dot{x} = 4\text{ m s}^{-1}\quad 4 = 8\cos 2t$$

$$\cos 2t = 0.5$$

$$t = \frac{1}{2}\cos^{-1}0.5$$

$$t = \frac{1}{2}\times\frac{\pi}{3}$$

The least value of t is $\frac{\pi}{6}$.

> From **a**.

13 e $x = 4\sin 2t$

$x = 2 \quad 2 = 4\sin 2t$

$\sin 2t = 0.5$

$t = \frac{1}{2}\sin^{-1} 0.5$

$t = \frac{1}{2} \times \frac{\pi}{6}$

The least value of t is $\frac{\pi}{12}$.

14 a $x = 3\sin\left(4t + \frac{1}{2}\right)$

$\dot{x} = 12\cos\left(4t + \frac{1}{2}\right)$

$\ddot{x} = -48\sin\left(4t + \frac{1}{2}\right)$

$\ddot{x} = -16x$

$\therefore$ S.H.M.

b amplitude = 3 m

period $= \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}$ s

Compare with $x = a\sin(\omega t + \varepsilon)$ to obtain a and ω.

c $t = 0 \quad x = 3\sin\left(\frac{1}{2}\right)$

$= 1.438...$

When $t = 0$, $x = 1.44$ (3 s.f.)

d $x = 0 \quad 0 = 3\sin\left(4t + \frac{1}{2}\right)$

$\sin\left(4t + \frac{1}{2}\right) = 0$

$4t + \frac{1}{2} = 0, \pi, ...$

$4t = \left(0 - \frac{1}{2}\right), \left(\pi - \frac{1}{2}\right), ...$

$t = -\frac{1}{8}$ (not applicable)

$t = \frac{1}{4}\left(\pi - \frac{1}{2}\right) = 0.6603...$

The value of t is 0.660 (3 s.f.).

15 a amplitude $= \frac{(15-5)}{2} = 5\,\text{m}$

The difference between high and low tides is twice the amplitude.

period $= (10\,\text{am} \to 4.15\,\text{pm}) \times 2$
$= 6.25 \times 2\,\text{hr}$
$= 12.5\,\text{hr}$

The time from low to high tide is half the period.

You can work through this question using hours as the unit for time.

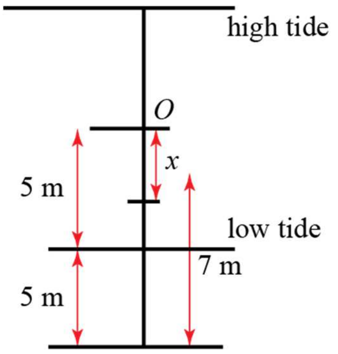
period $= \frac{2\pi}{\omega} = 12.5$

$\omega = \frac{2\pi}{12.5}$

$x = a\cos\omega t$

Start at low tide.

$x = 5\cos\left(\frac{2\pi}{12.5}t\right)$



$x = 3\,\text{m}$

$3 = 5\cos\left(\frac{2\pi}{12.5}t\right)$

$\cos\left(\frac{2\pi}{12.5}t\right) = 0.6$

$t = \frac{12.5}{2\pi}\cos^{-1}0.6$

$t = 1.844...$

The diagram shows that when the water is 7 m deep, $x = 3$.

Time after 10 am.

The ship can enter the harbour at 11.51 am. (nearest minute).

b The water is once more 7 m deep at (12.5–1.844) hours after 10 am

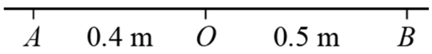
= 10.656 hrs after 10 am

= 10 hr 39.3...min.

∴ Ship must leave by 8.39 pm (nearest minute).

Remember to change the decimal part of 1.844 into minutes (0.844×60).

Use the symmetry of S.H.M. to find the time required.

16 

$$\text{period} = \frac{2\pi}{\omega} = 4$$

$$\omega = \frac{\pi}{2}$$

$$x = a\sin\omega t$$

$$x = 0.75\sin\frac{\pi}{2}t$$

> Find the time taken from O to B (using $x = 0.5$ m) and from O to the point where $x = 0.4$ m.

$x = 0.5\text{m} \qquad 0.5 = 0.75\sin\frac{\pi}{2}t$

$$\sin\frac{\pi t}{2} = \frac{0.5}{0.75}$$

$$t = \frac{2}{\pi}\sin^{-1}\left(\frac{0.5}{0.75}\right)$$

$x = 0.4\,\text{m} \qquad t = \frac{2}{\pi}\sin^{-1}\left(\frac{0.4}{0.75}\right)$

Time $B \to A$

$$= \frac{2}{\pi}\left[\sin^{-1}\left(\frac{0.5}{0.75}\right) + \sin^{-1}\left(\frac{0.4}{0.75}\right)\right]$$

$$= 0.8226...$$

> Adding these times will give the time to go directly from B to A due to the symmetry of S.H.M.

P takes 0.823s to travel directly from B to A (3 s.f.)

Challenge

$\ddot{x} = -\omega^2 x \qquad v^2 = \omega^2\left(a^2 - x^2\right)$

$v_1^2 = \omega^2\left(a^2 - x_1^2\right)$ **(1)**

$v_2^2 = \omega^2\left(a^2 - x_2^2\right)$ **(2)**

(2) − **(1)**: $v_2^2 - v_1^2 = \omega^2(a^2 - x_2^2) - \omega^2(a^2 - x_1^2)$

$v_2^2 - v_1^2 = \omega^2(a^2 - x_2^2 - a^2 + x_1^2)$

Rearranging gives $\omega^2 = \frac{v_2^2 - v_1^2}{x_1^2 - x_2^2}$ so $\omega^2 = \left(\frac{v_2^2 - v_1^2}{x_1^2 - x_2^2}\right)^{\frac{1}{2}}$

$$T = \frac{2\pi}{\omega} = 2\pi\left(\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}\right)^{\frac{1}{2}}$$